Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $\mathcal{M}$ be the set of all $n \times n$ real matrices. Any matrix $f \in \mathcal{M}$ can be seen as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.

   (a) Show that any $f, g \in \mathcal{M}$ are homotopic.

   (b) Is the homotopy between f, g above only involves mappings in $\mathcal{M}$? That is, there exists a homotopy $H \colon \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ between f, g such that for each $t \in [0, 1]$, the mapping $x \mapsto H(x, t)$ also belongs to $\mathcal{M}$. We call it a homotopy through mappings in $\mathcal{M}$.

   (c) Let $\mathcal{A} \subset \mathcal{M}$ be the subset of invertible matrices and $f, g \in \mathcal{A}$. Are they homotopic through mappings in $\mathcal{A}$?

   (d) If $f, g \in \mathcal{P}$, the set of positive definite matrices, then there is a homotopy between f and g through mappings in $\mathcal{P}$.

2. Let $\mathcal{M}$ be the set of all $n \times n$ real matrices. It can be given a topology induced by the standard $\mathbb{R}^{n^2}$. Show that $\mathcal{M}$ is path connected if and only if every pair of $f, g \in \mathcal{M}$ are homotopic through mappings in $\mathcal{M}$.

3. If $f_1 \simeq g_1 \colon X \to Y_1$ and $f_2 \simeq g_2 \colon X \to Y_2$, show that $(f_1, f_2) \simeq (g_1, g_2)$ as mappings $X \to (Y_1 \times Y_2)$, where $(f_1, f_2)(x) = (f_1(x), f_2(x))$ and $(g_1, g_2)(x) = (g_1(x), g_2(x))$.

4. Let $\mathbb{D} = \{ z \in \mathbb{C} \colon |z| \le 1 \}$ and $f \colon \mathbb{D} \to \mathbb{D}$ be given by $f(z) = |z|\, e^{2\pi |z| \mathbf{i}}$. Show that f is homotopic to the identity mapping on $\mathbb{D}$. Geometrically visualize the action.

   Note that f is indeed a homeomorphism. Can you find a homotopy H such that at every $t \in [0, 1]$, the map $z \mapsto H(z, t)$ is also a homeomorphism on $\mathbb{D}$?

5. Let $A = \{ z \in \mathbb{C} \colon 1 \le |z| \le 2 \}$ and $f \colon \mathbb{D} \to \mathbb{D}$ be a homeomorphism on A given by $f(z) = |z|\, e^{2\pi(|z|-1)\mathbf{i}}$. Is it homotopy to the identity mapping on A?

6. Let $f, g \colon X \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be two mappings such that for all $x \in X$, $f(x) \ne -g(x)$. Show that $(1-t)f(x) + tg(x)$ will give a homotopy between f and g.

7. Show that if $f \colon X \to \mathbb{S}^n$ is not surjective, then f is null homotopic.

8. Let $a\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the antipodal map, i.e., $a(z) = -z$. Show that $a \simeq \mathrm{id}_{\mathbb{S}^1}$. Note that this is not true for $\mathbb{S}^n$ with even n but it holds for odd n.

9. Let Y be any topological space. Form the quotient space $CY = (Y \times [0,1])/\sim$ by the equivalence relation $\sim$ on $Y \times [0,1]$ with $(y_1, t_1) \sim (y_2, t_2)$ if $t_1 = 1 = t_2$. That is, CY is obtained by crushing the "top" $Y \times \{1\}$ to one point. Prove that any map $f\colon X \to CY$ is null homotopic.

10. Show that a map $f\colon X \to Y$ is null homotopic if and only if there exists $\tilde{f}\colon CX \to Y$ such that $\tilde{f}|_X \equiv f$ by naturally seeing $X \hookrightarrow CX$ as a subspace.

11. Show that homotopy equivalence (homotopy type) defines an equivalence relation on all the topological spaces.

12. Show that a space of two points, i.e., $X = \{-1, 1\}$ with discrete topology, is not homotopy equivalent to a one point space. In other words, X is not contractible.

13. Try to convince yourself that $\mathbb{S}^n$ is not contractible (the rigorous proof may be beyond your knowledge now, see exercise below).

    Remark. Note that the two-point space above is defined as the 0-dimensional sphere, $\mathbb{S}^0$.

14. Consider the unit sphere $\mathbb{S}^{n-1}$ and the punctured space $\mathbb{R}^n \setminus \{0\}$. Show that they are homotopy equivalent. In fact, $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

15. Give explicit argument of why $\mathbb{S}^1 \vee \mathbb{S}^1$ is a deformation retract of a one-punctured torus.

16. Given $f\colon X \to Y$, there is a natural mapping, again denote it by f, from $X \times \{0\} \to Y$. One may define the quotient spaces (called mapping cylinder and mapping cone),

$$M_f = ((X \times [0,1]) \coprod Y)/ \sim, \quad \text{where } (x, 0) \sim f(x);$$
$$C_f = ((X \times [0,1]) \coprod Y)/ \sim, \quad \text{where } (x, 0) \sim f(x) \text{ and } (x_1, 1) \sim (x_2, 1).$$

    Remark. To understand them, imagine $f\colon \mathbb{S}^1 \to \mathbb{R}$ to be the standard embedding. Then M_f is a tall hat while C_f is a wizard hat. In general, f need not to be one-to-one. In addition, if $\mathbb{D}^n$ is the closed n-dimensional unit disk and $f\colon \mathbb{S}^n \to \mathbb{D}^{n+1}$ is the standard embedding, then $C_f = \mathbb{S}^{n+1}$.

    Show that if $f, g\colon X \to Y$ are homotopic mappings, then M_f and M_g are homotopy equivalent; likewise, C_f and C_g are also homotopy equivalent.

    Remark. Using this, one may prove the USELESS result: if $\mathbb{S}^n$ is contractible then so is $\mathbb{S}^{n+1}$. The converse is the USEFUL part because one may set up an induction process. Together with that $\mathbb{S}^0$ is not contractible (done above), we prove $\mathbb{S}^n$ is not contractible.

    For those who are interested, you may try to show if both X and Y are Hausdorff, then so are M_f and C_f.